Robot Dynamics

Kinematics vs. Dynamics

- Kinematics
  - Equations about position, velocity, and acceleration

- Dynamics
  - Equations about force and/or torque
  - Applications
    1) Control problem

$$\Theta, \dot{\Theta}, \ddot{\Theta} \quad \Rightarrow \quad \tau$$

    2) Simulation problem

$$\tau \quad \Rightarrow \Theta, \dot{\Theta}, \ddot{\Theta}$$

Force and Torque

- Torque (회전력)

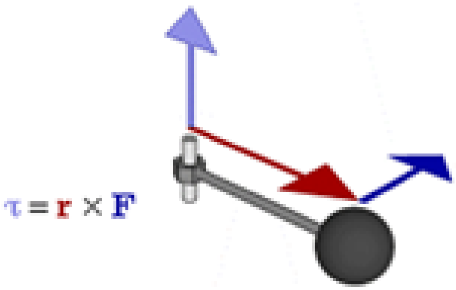
$\vec{\tau} = \vec{r} \times \vec{F}$ (Nm or Kg m^2 / sec^2)

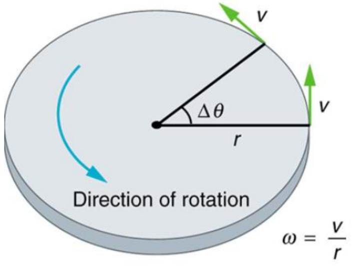
$$F = m\,a$$

⇨ $\tau = r \cdot ma = r \cdot m \cdot r\alpha$ (α : angular acceleration)

$= mr^2 \cdot \alpha$

$= I\,\alpha$ (I : inertia, 관성질량, Kg m^2)





$$\alpha = \dot{\omega}$$

- Kinetic Energy

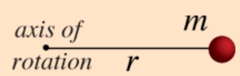
$$K_{translational} = \frac{1}{2}mv^2$$

$$K_{rotational} = \frac{1}{2}I\omega^2$$

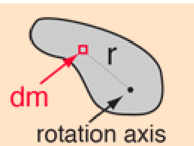
Force and Torque

- Inertia Calculation

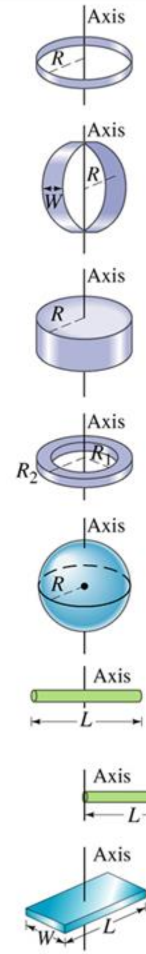
$$I = mr^2$$



$$I = \int dI = \int_0^M r^2 dm$$

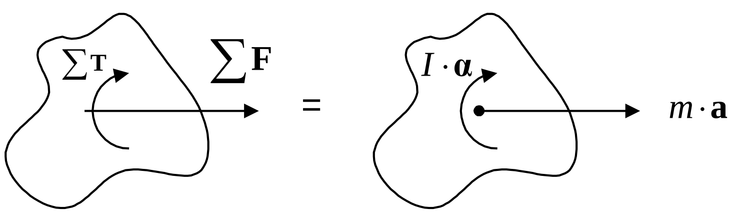


	Object	Location of axis	Moment of inertia
(a)	**Thin hoop,** radius R	Through center	MR^2
(b)	**Thin hoop,** radius R width W	Through central diameter	$\frac{1}{2}MR^2 + \frac{1}{12}MW^2$
(c)	**Solid cylinder,** radius R	Through center	$\frac{1}{2}MR^2$
(d)	**Hollow cylinder,** inner radius R_1 outer radius R_2	Through center	$\frac{1}{2}M(R_1^2 + R_2^2)$
(e)	**Uniform sphere,** radius R	Through center	$\frac{2}{5}MR^2$
(f)	**Long uniform rod,** length L	Through center	$\frac{1}{12}ML^2$
(g)	**Long uniform rod,** length L	Through end	$\frac{1}{3}ML^2$
(h)	**Rectangular thin plate,** length L, width W	Through center	$\frac{1}{12}M(L^2 + W^2)$



Dynamics

Dynamics	Newton-Euler	Lagrangian
Approach	Force balance equation $\sum \mathbf{F} = m \cdot \mathbf{a}$ and $\sum \mathbf{T} = I \cdot \boldsymbol{\alpha}$	Energy balance equation $L = K - P$ $F_i = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i}$ $T_i = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i}$
Form	Iterative	Closed
Use	Easier for simpler systems	Easier for more complicated systems



Lagrangian Mechanics

- Lagrangian

$L = K - P$

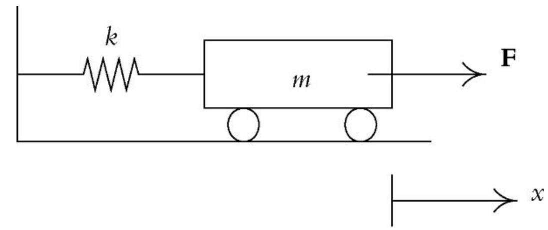
L : Lagrangian, K : Kinetic energy, P: potential energy

- Lagrangian relationships

$$F_i = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} \qquad T_i = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i}$$

Lagrangian Mechanics

(Ex1) 1 d.o.f system



- Kinetic energy

$$K = \frac{1}{2}\, m\, v^2 = \frac{1}{2}\, m\, \dot{x}^2$$

- Potential energy

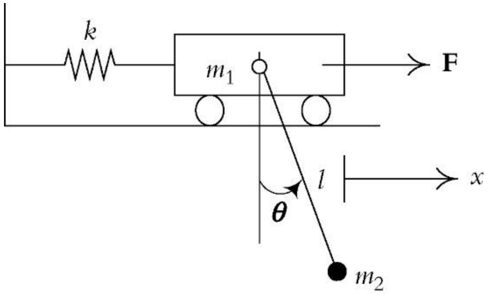
$$P = \frac{1}{2}\, k\, x^2$$

⇨ $$L = K - P = \frac{1}{2}\, m\, \dot{x}^2 - \frac{1}{2}\, k\, x^2$$

⇨ $$F = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \frac{\partial}{\partial t}(m\dot{x}) - (-kx) = m\ddot{x} + kx$$

Lagrangian Mechanics

(Ex2) 2 d.o.f system



- Kinetic energy

$$K = K_{cart} + K_{pendulum} = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2 v_{pendulum}^2$$

$$\vec{V}_{pendulum} = \vec{V}_c + \vec{V}_{plc} = \dot{x}\,\hat{\imath} + l\,\dot{\theta}\cos\theta\,\hat{\imath} + l\,\dot{\theta}\sin\theta\,\hat{\jmath} = \left(\dot{x} + l\,\dot{\theta}\cos\theta\right)\hat{\imath} + l\dot{\theta}\sin\theta\,\hat{\jmath}$$

$$v_{pendulum}^2 = \left(\dot{x} + l\,\dot{\theta}\cos\theta\right)^2 + (l\dot{\theta}\sin\theta)^2$$

⇨ $$K = \frac{1}{2}(m_1 + m_2)\,\dot{x}^2 + \frac{1}{2}\,m_2(l^2\dot{\theta}^2 + 2l\dot{\theta}\dot{x}\cos\theta)$$

- Potential energy

$$P = \frac{1}{2}\,k\,x^2 + m_2 g l(1 - \cos\theta)$$

⇨ $$L = K - P = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}\,m_2\left(l^2\dot{\theta}^2 + 2l\dot{\theta}\dot{x}\cos\theta\right) - \frac{1}{2}\,k\,x^2 - m_2 g l(1 - \cos\theta)$$

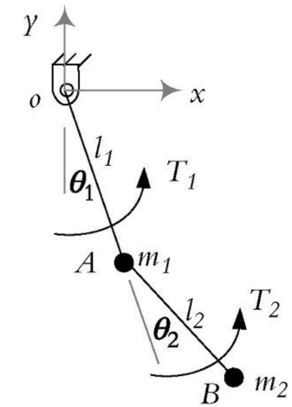
⇨ $$F = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = (m_1 + m_2)\ddot{x} + m_2 l\ddot{\theta}\cos\theta - m_2 l\dot{\theta}^2\cos\theta + kx$$

$$T = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = m_2 l^2\ddot{\theta} + m_2 l\ddot{x}\cos\theta + m_2 g l\sin\theta$$

⇨ $$\begin{bmatrix} F \\ T \end{bmatrix} = \begin{bmatrix} m_1 + m_2 & m_2 l\cos\theta \\ m_2 l\cos\theta & m_2 l^2 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & m_2 l\sin\theta \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}^2 \\ \dot{\theta}^2 \end{bmatrix} + \begin{bmatrix} kx \\ m_2 g l\sin\theta \end{bmatrix}$$

Lagrangian Mechanics

(Ex3) 2 d.o.f system with point mass



- Kinetic energy $K = K_1 + K_2$

$$K_1 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2$$

$$K_2 = \frac{1}{2} m_2 V_2^2$$

$$\begin{cases} x_2 = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) = l_1 S_1 + l_2 S_{12} \\ y_2 = -l_1 C_1 - l_2 C_{12} \end{cases}$$

$$\begin{cases} \dot{x}_2 = l_1 C_1 \dot{\theta}_1 + l_2 C_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ \dot{y}_2 = l_1 S_1 \dot{\theta}_1 + l_2 S_{12}(\dot{\theta}_1 + \dot{\theta}_2) \end{cases}$$

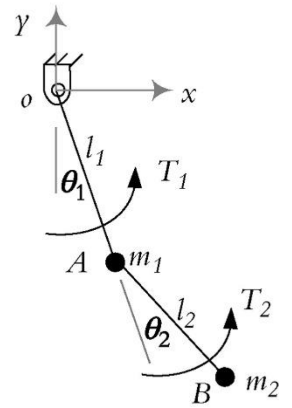
⇨ $V_2 = \dot{x}_2^2 + y_2^2$

⇨
$$\begin{aligned} V_2^2 &= l_1^2 \dot{\theta}_1^2 + l_2^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2) + 2 l_1 l_2 (\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2)(C_1 C_{12} + S_1 S_{12}) \\ &= l_1^2 \dot{\theta}_2^2 + l_2^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2) + 2 l_1 l_2 C_2 (\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) \end{aligned}$$

⇨
$$K_2 = \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2) + m_2 l_1 l_2 C_2 (\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2)$$

⇨
$$K = \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta}_2 + \frac{1}{2} m_2 l_2^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2) + m_2 l_1 l_2 C_2 (\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2)$$

Lagrangian Mechanics



- Potential Energy $P = P_1 + P_2$

$$P_1 = -m_1 g l_1 C_1$$
$$P_2 = -m_2 g l_1 C_1 - m_2 g l_2 C_{12}$$
$$P = P_1 + P_2 = -(m_1 + m_2) g l_1 C_1 - m_2 g l_2 C_{12}$$

- Lagrangian

$$L = K - P = \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2) + m_2 l_1 l_2 C_2 (\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) + (m_1 + m_2) g l_1 C_1 + m_2 g l_2 C_{12}$$

- Dynamic equation

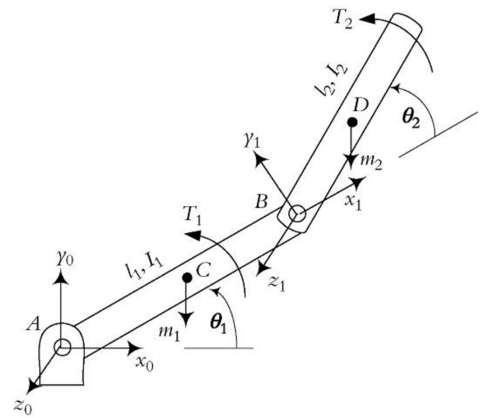
$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} (m_1 + m_2) l_1^2 + m_2 l_2^2 + 2 m_2 l_1 l_2 C_2 & m_2 l_2^2 + m_2 l_1 l_2 C_2 \\ (m_2 l_2^2 + m_2 l_1 l_2 C_2) & m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 & -m_2 l_1 l_2 S_2 \\ m_2 l_1 l_2 S_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} -m_2 l_1 l_2 S_2 & -m_2 l_1 l_2 S_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \dot{\theta}_2 \\ \dot{\theta}_2 \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} (m_1 + m_2) g l_1 S_1 + m_2 g l_2 S_{12} \\ m_2 g l_2 S_{12} \end{bmatrix}$$

Lagrangian Mechanics

(Ex4) 2 link arm

- Kinetic energy

$$K = K_1 + K_2 = \left[\frac{1}{2} I_A \dot{\theta}_1^2\right] + \left[\frac{1}{2} I_D(\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{1}{2} m_2 V_D^2\right]$$
$$= \left[\frac{1}{2}\left(\frac{1}{3} m_1 l_1^2\right)\dot{\theta}_1^2\right] + \left[\frac{1}{2}\left(\frac{1}{12} m_2 l_2^2\right)(\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{1}{2} m_2 V_D^2\right]$$



$$x_D = l_1 C_1 + 0.5 l_2 C_{12} \quad \rightarrow \quad \dot{x}_D = -l_1 S_1 \dot{\theta}_1 - 0.5 l_2 S_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$
$$y_D = l_1 S_1 + 0.5 l_2 S_{12} \quad \rightarrow \quad \dot{y}_D = l_1 C_1 \dot{\theta}_1 + 0.5 l_2 C_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$

⇨

$$V_D^2 = \dot{x}_D^2 + \dot{y}_D^2$$
$$= \dot{\theta}_1^2 (l_1^2 + 0.25 l_2^2 + l_1 l_2 C_2) + \dot{\theta}_2^2 (0.25 l_2^2) + \dot{\theta}_1 \dot{\theta}_2 (0.5 l_2^2 + l_1 l_2 C_2)$$

⇨

$$K = \dot{\theta}_1^2 \left(\frac{1}{6} m_1 l_1^2 + \frac{1}{6} m_2 l_2^2 + \frac{1}{2} m_2 l_1{}^2 + \frac{1}{2} m_2 l_1 l_2 C_2\right)$$
$$+ \dot{\theta}_2^2 \left(\frac{1}{6} m_2 l_2^2\right) + \dot{\theta}_1 \dot{\theta}_2 \left(\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2\right)$$

Lagrangian Mechanics

- Potential Energy

$$P = m_1 g \frac{l_1}{2} S_1 + m_2 g \left(l_1 S_1 + \frac{l_2}{2} S_{12} \right)$$

- Lagrangian

$$L = K - P = \dot{\theta}_1^2 \left(\frac{1}{6} m_1 l_1^2 + \frac{1}{6} m_2 l_2^2 + \frac{1}{2} m_2 l_1^2 + \frac{1}{2} m_2 l_1 l_2 C_2 \right) + \dot{\theta}_2^2 \left(\frac{1}{6} m_2 l_2^2 \right)$$
$$+ \dot{\theta}_1 \dot{\theta}_2 \left(\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2 \right) - m_1 g \frac{l_1}{2} S_1 - m_2 g \left(l_1 S_1 + \frac{l_2}{2} S_{12} \right)$$

- Dynamic equation

$$T_1 = \left(\frac{1}{3} m_1 l_1^2 + m_2 l_1^2 + \frac{1}{3} m_2 l_2^2 + m_2 l_1 l_2 C_2 \right) \ddot{\theta}_1 + \left(\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2 \right) \ddot{\theta}_2$$
$$- (m_2 l_1 l_2 S_2) \dot{\theta}_1 \dot{\theta}_2 - \left(\frac{1}{2} m_2 l_1 l_2 S_2 \right) \dot{\theta}_2^2$$
$$+ \left(\frac{1}{2} m_1 + m_2 \right) g l_1 C_1 + \frac{1}{2} m_2 g l_2 C_{12}$$

$$T_2 = \left(\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2 \right) \ddot{\theta}_1 + \left(\frac{1}{3} m_2 l_2^2 \right) \ddot{\theta}_2 + \left(\frac{1}{2} m_2 l_1 l_2 S_2 \right) \dot{\theta}_1^2 + \frac{1}{2} m_2 g l_2 C_{12}$$

Robot Dynamic Equations

- 2 d.o.f system

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} D_{ii} & D_{ij} \\ D_{ji} & D_{jj} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_i \\ \ddot{\theta}_j \end{bmatrix} + \begin{bmatrix} D_{iii} & D_{ijj} \\ D_{jii} & D_{jjj} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} = \begin{bmatrix} D_{iij} & D_{iji} \\ D_{jij} & D_{jji} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \dot{\theta}_2 \\ \dot{\theta}_2 \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} D_i \\ D_j \end{bmatrix}$$

Inertia Centrifugal Coriolis Gravity

- N d.o.f system

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

$M(\Theta)$: inertia matrix (n x n)
$V(\Theta, \dot{\Theta})$: vector of centrifugal and Coriolis (n x 1)
$G(\Theta)$: vector of gravity (n x 1)

Forces of Friction

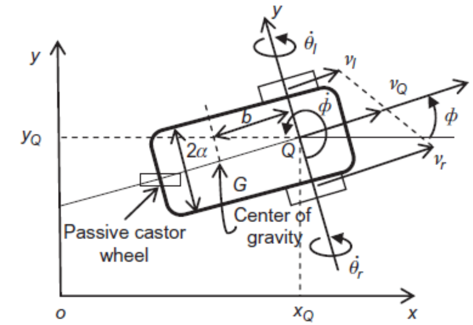
- Friction
  - Viscous friction $\tau_{friction} = v\dot{\theta}$
  - Coulomb friction $\tau_{friction} = c\ sgn(\dot{\theta})$

$$\tau_{friction} = c\ sgn(\dot{\theta}) + v\dot{\theta} \quad = f(\theta, \dot{\theta})$$

$$\boxed{\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta) + F(\Theta, \dot{\Theta})}$$

Differential Drive WMR

- Kinetic energy



$$K = K_1 + K_2 + K_3$$

$$K_1 = \frac{1}{2} m v_G^2 = \frac{1}{2} m (\dot{x}_G^2 + \dot{y}_G^2)$$

$$K_2 = \frac{1}{2} I_Q \dot{\phi}^2$$

$$K_3 = \frac{1}{2} I_o \dot{\theta}_r^2 + \frac{1}{2} I_o \dot{\theta}_l^2$$

$$\dot{x}_G = \dot{x}_Q + b\dot{\phi} \sin\phi$$

$$\dot{y}_G = \dot{y}_Q - b\dot{\phi} \cos\phi$$

$$\dot{x}_Q = \frac{r}{2}(\dot{\theta}_r \cos\phi + \dot{\theta}_l \cos\phi) = \frac{r}{2}(\dot{\theta}_r + \dot{\theta}_l)\cos\phi$$

$$\dot{y}_Q = \frac{r}{2}(\dot{\theta}_r \sin\phi + \dot{\theta}_l \sin\phi) = \frac{r}{2}(\dot{\theta}_r + \dot{\theta}_l)\sin\phi$$

$$\dot{\phi} = \frac{r}{2a}(\dot{\theta}_r - \dot{\theta}_l)$$

⇨

$$K(\dot{\theta}_r, \dot{\theta}_l) = \left[\frac{mr^2}{8} + \frac{(I_Q + mb^2)r^2}{8a^2} + \frac{I_o}{2}\right]\dot{\theta}_r^2 + \left[\frac{mr^2}{8} + \frac{(I_Q + mb^2)r^2}{8a^2} + \frac{I_o}{2}\right]\dot{\theta}_l^2 + \left[\frac{mr^2}{4} - \frac{(I_Q + mb^2)r^2}{4a^2}\right]\dot{\theta}_r\dot{\theta}_l$$

m = mass of the entire robot
v_G = linear velocity of the COG G
I_Q = moment of inertia of the robot with respect to Q
I_o = moment of inertia of each wheel plus the corresponding motor's rotor moment of inertia.

Differential Drive WMR

- Lagrangian

$$L = K \qquad \text{(Potential energy = 0)}$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}_r}\right) - \frac{\partial K}{\partial \theta_r} = \tau_r - \beta\dot{\theta}_r$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}_l}\right) - \frac{\partial K}{\partial \theta_l} = \tau_l - \beta\dot{\theta}_l$$

β is the wheels' common friction coefficient

⇨

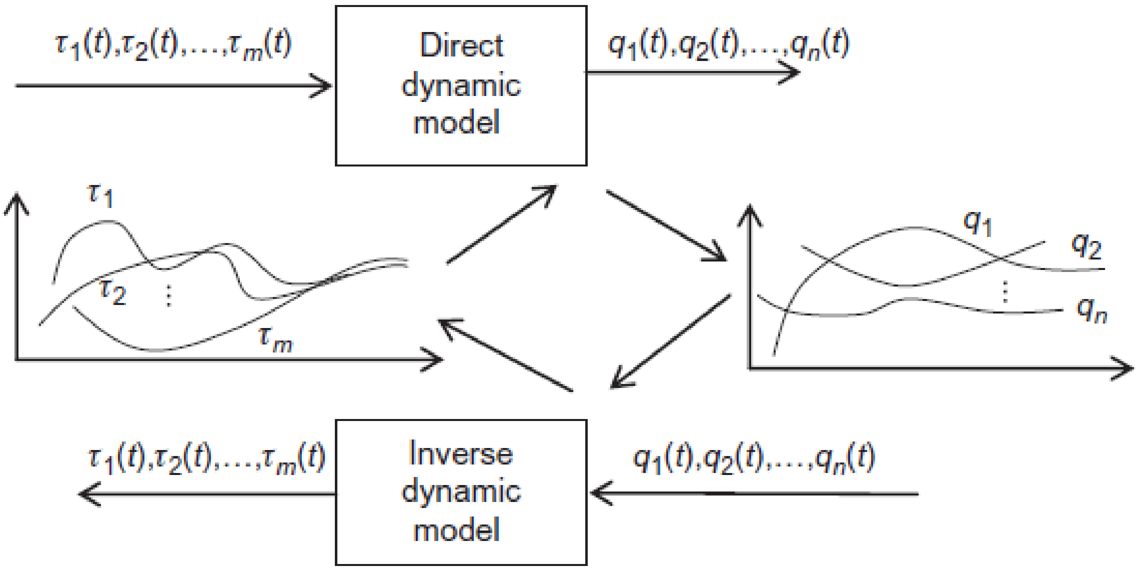
$$D_{11}\ddot{\theta}_r + D_{12}\ddot{\theta}_l + \beta\dot{\theta}_r = \tau_r$$

$$D_{21}\ddot{\theta}_r + D_{22}\ddot{\theta}_l + \beta\dot{\theta}_l = \tau_l$$

$$D_{11} = D_{22} = \left[\frac{mr^2}{4} + \frac{(I_Q + mb^2)r^2}{8a^2} + I_o\right]$$

$$D_{12} = D_{21} = \left[\frac{mr^2}{4} - \frac{(I_Q + mb^2)r^2}{8a^2}\right]$$

Dynamic Simulation



Dynamic Simulation

- Motion model

$$\ddot{\Theta} = M^{-1}(\Theta)[\tau - V(\Theta, \dot{\Theta}) - G(\Theta) - F(\Theta, \dot{\Theta})] \qquad (\bigstar)$$

Given: torque profile $\tau(t)$, initial position & velocity $\Theta(0)$, $\dot{\Theta}(0)$
Find: position & velocity profile $\Theta(t)$, $\dot{\Theta}(t)$

$$\Theta(0) = \Theta_0,$$

$$\dot{\Theta}(0) = 0,$$

$$\ddot{\Theta}(0) \qquad (\bigstar)$$

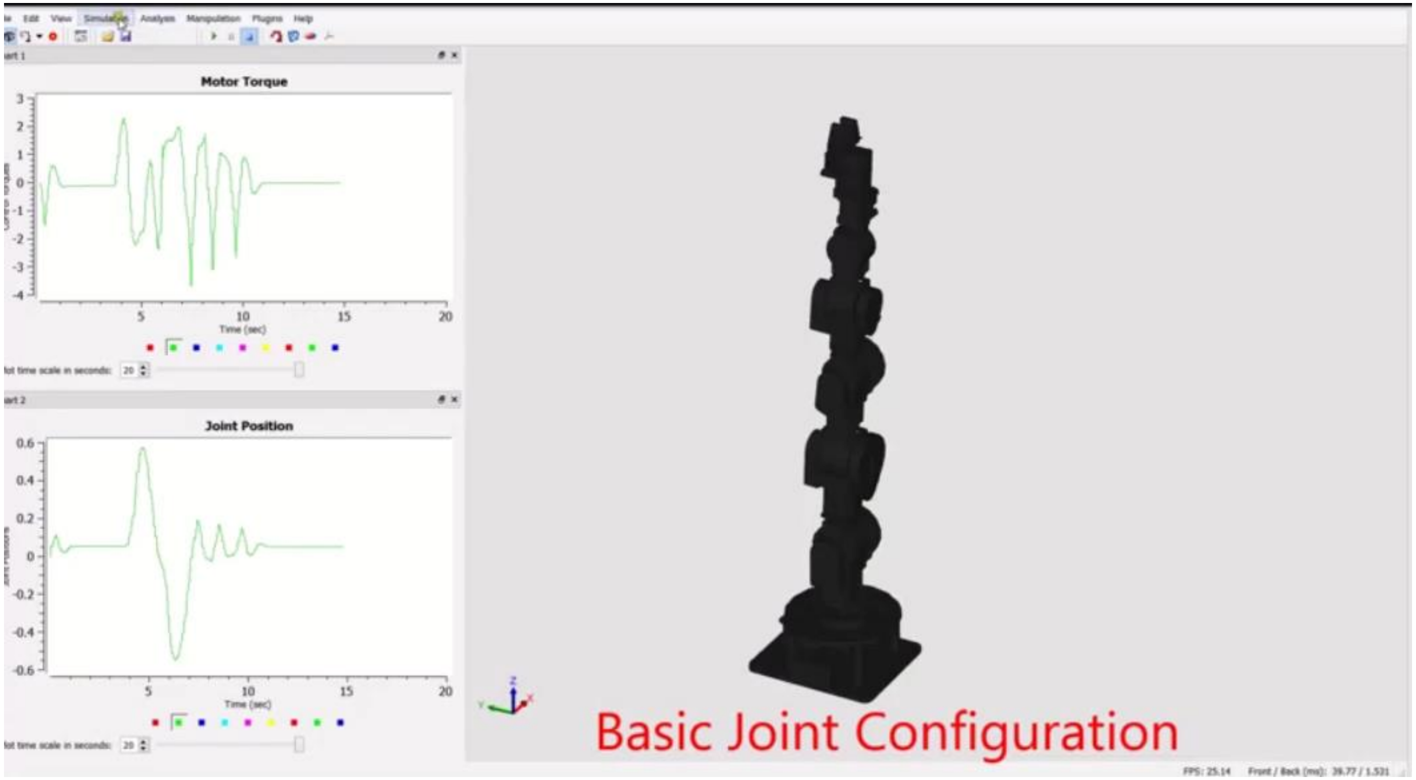
$$\Rightarrow \quad \dot{\Theta}(t + \Delta t) = \dot{\Theta}(t) + \ddot{\Theta}(t)\Delta t,$$

$$\Theta(t + \Delta t) = \Theta(t) + \dot{\Theta}(t)\Delta t + \tfrac{1}{2}\ddot{\Theta}(t)\Delta t^2,$$

(Euler integration)

$$\ddot{\Theta}(t + \Delta t) \qquad (\bigstar)$$

Dynamic Simulation



Static Force Analysis

- Hand force vs. Joint torque

$$\left[{}^{H}F\right]=\left[f_x \quad f_y \quad f_z \quad m_x \quad m_y \quad m_z\right]^T$$: Force/moment at the hand

$$\left[{}^{H}D\right]=\left[dx \quad dy \quad dz \quad \delta x \quad \delta y \quad \delta z\right]^T$$: Displacements at the hand

$$\left[T\right]=\left[T_1 \quad T_2 \quad T_3 \quad T_4 \quad T_5 \quad T_6\right]^T$$: Torques at the joints

$$\left[D_\theta\right]=\left[d\theta_1 \quad d\theta_2 \quad d\theta_3 \quad d\theta_4 \quad d\theta_5 \quad d\theta_6\right]^T$$: Displacements at the joints

Static Force Analysis

- Energy (work) conservation

$$\delta W = \left[{}^{H}F\right]^{T}\left[{}^{H}D\right] = [T]^{T}[D_{\theta}] \Leftrightarrow \begin{bmatrix} f_x & f_y & f_z & m_x & m_y & m_z \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \\ \delta x \\ \delta y \\ \delta z \end{bmatrix} = f_x dx + f_y dy + \cdots + m_z \delta z$$

- Jacobian

$$\left[{}^{T_6}D\right] = \left[{}^{T_6}J\right][D_{\theta}] \qquad \left[{}^{H}D\right] = \left[{}^{H}J\right][D_{\theta}]$$

$$\Rightarrow \left[{}^{H}F\right]^{T}\left[{}^{H}J\right][D_{\theta}] = [T]^{T}[D_{\theta}] \rightarrow \left[{}^{H}F\right]^{T}\left[{}^{H}J\right] = [T]^{T}$$

$$\Rightarrow \boxed{[T] = \left[{}^{H}J\right]^{T}\left[{}^{H}F\right]}$$

Static Force Analysis

(Ex) 구형-RPY 로봇(예를 들어 스탠포드 암)의 자코비안 수치값이 다음과 같이 주어져 있다. 손의 좌표계의 z축을 따라 1 [lb]의 힘이 실리며 블록에 구멍을 뚫기 위해서 z축을 따라 20 [lb · in]의 모멘트가 작용한다. 필요한 관절의 힘과 토크를 구하라.

$$
{}^{H}J = \begin{bmatrix} 20 & 0 & 0 & 0 & 0 & 0 \\ -5 & 0 & 1 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$