# Position Analysis

# Robot as Mechanisms

- Manipulator-type robots
  = multiple DOF, 3 dimensional, open-loop mechanisms

- Closed-loop vs. Open-loop mechanism

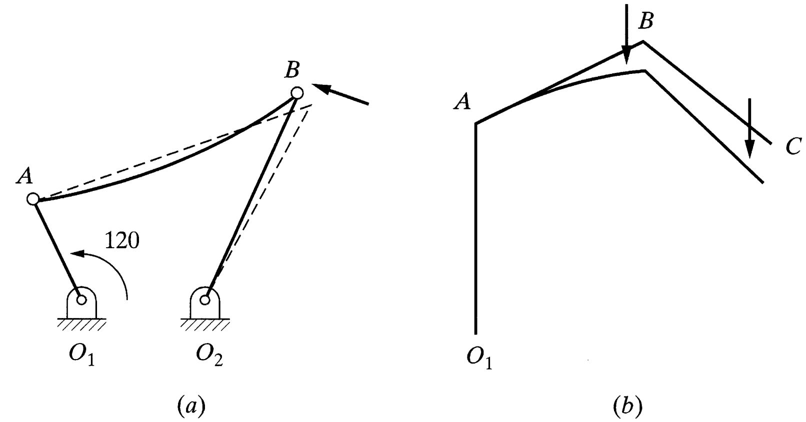



Fig. 2.2 (a) Closed-loop versus (b) open-loop mechanism

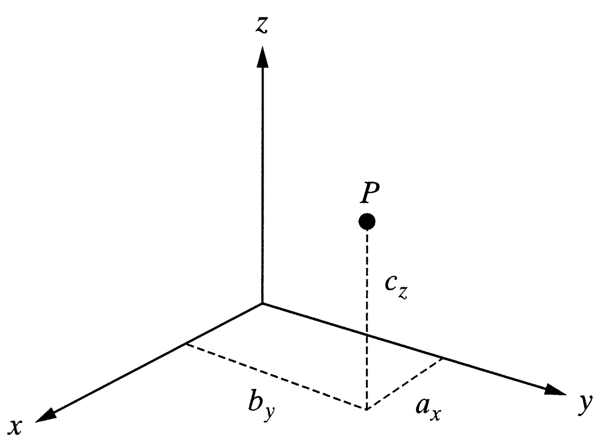
**Figure 2.3** Possible parallel manipulator configurations.

closed-loop → parallel manipulator
open-loop → serial manipulator

# Matrix Representation

- **Cartesian Frame**
  - x,y,z axes: 서로 직교(orthogonal), 오른나사법칙
  - 각도기준방향: (+) CCW   (-) CW

- A point *P* in space :
  - 3 coordinates relative to a reference frame




$$P = a_x \hat{i} + b_y \hat{j} + c_z \hat{k}$$

Fig. 2.3 Representation of a point in space

# Matrix Representation

- A Vector *P* in space :
  - 3 x 1 matrix
  - 4 x 1 matrix

$$\overline{P} = a_x \hat{i} + b_y \hat{j} + c_z \hat{k}$$

$$\overline{P} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, a_x = \frac{x}{w}, b_y = \frac{y}{w}, c_z = \frac{z}{w}$$

$w$ : scale factor

$w = 0$ : direction vector

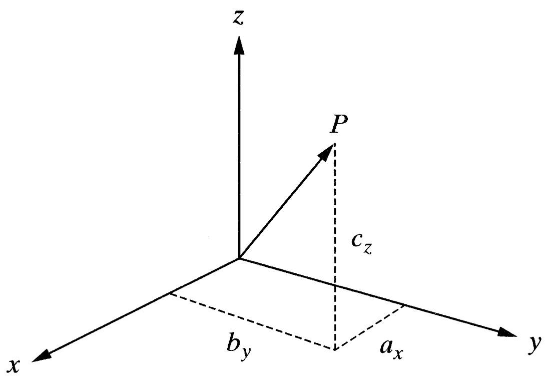



Fig. 2.4 Representation of a vector in space

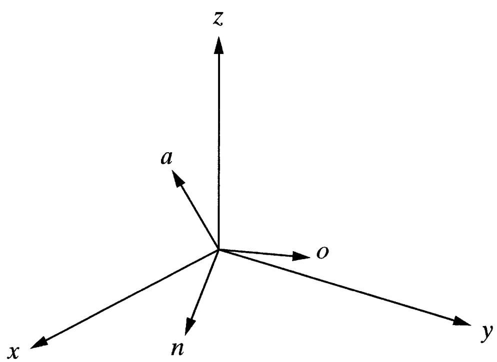
# Examples

(Ex) $\mathrm{P} = 3\hat{\imath} + 5\hat{\jmath} + 2\hat{k}$

(Ex) $q_{unit} = \begin{bmatrix} 0.371 \\ 0.557 \\ q_z \\ 0 \end{bmatrix}$

# Matrix Representation

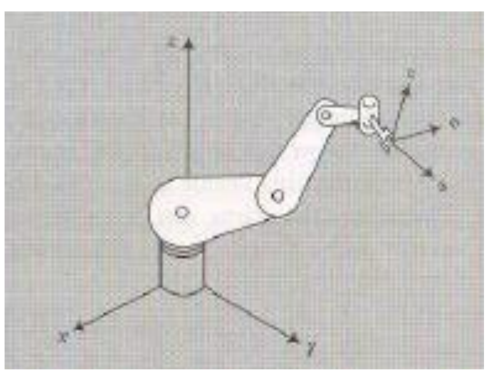
- Representation of a frame at the origin of a fixed-reference frame
  - 3 x 3 matrix
  - each unit vector (n, o, a) is mutually perpendicular




$$F = \begin{bmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix} = \begin{bmatrix} \bar{n} & \bar{o} & \bar{a} \end{bmatrix}$$

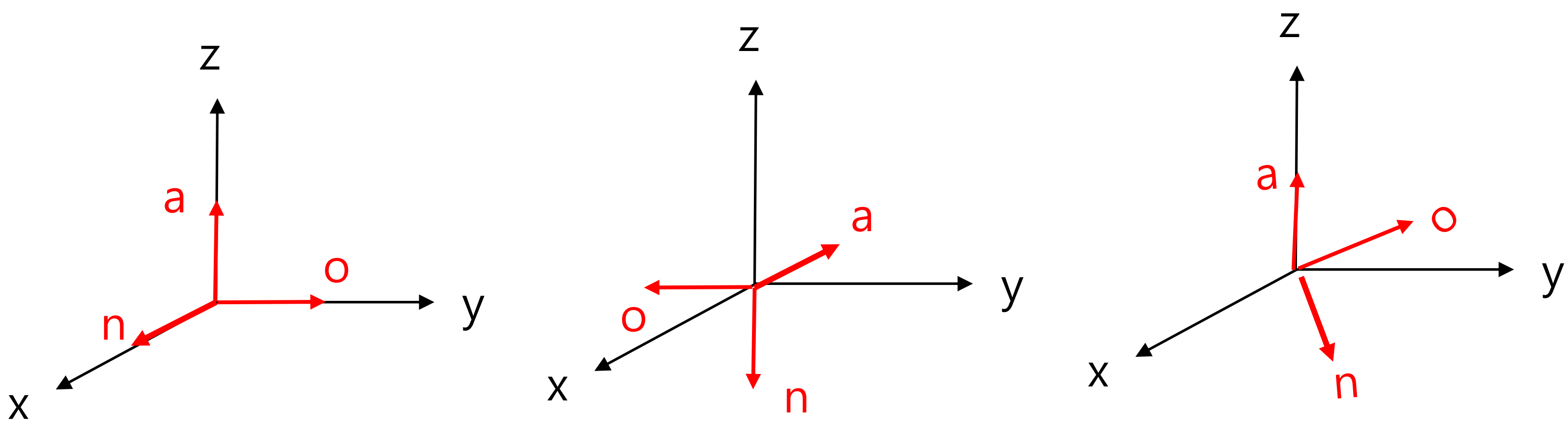
$$\|\bar{n}\| = \|\bar{o}\| = \|\bar{a}\| = 1$$

$$\bar{n} \cdot \bar{o} = \bar{o} \cdot \bar{a} = \bar{a} \cdot \bar{n} = 0$$



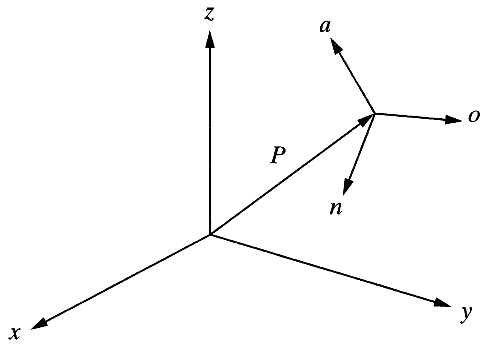
n: normal
o: orientation
a: approach

# Examples



# Matrix Representation

- Representation of a Frame in a Fixed Reference Frame
  - 4 x 4 matrix
  - (n, o, a, p)

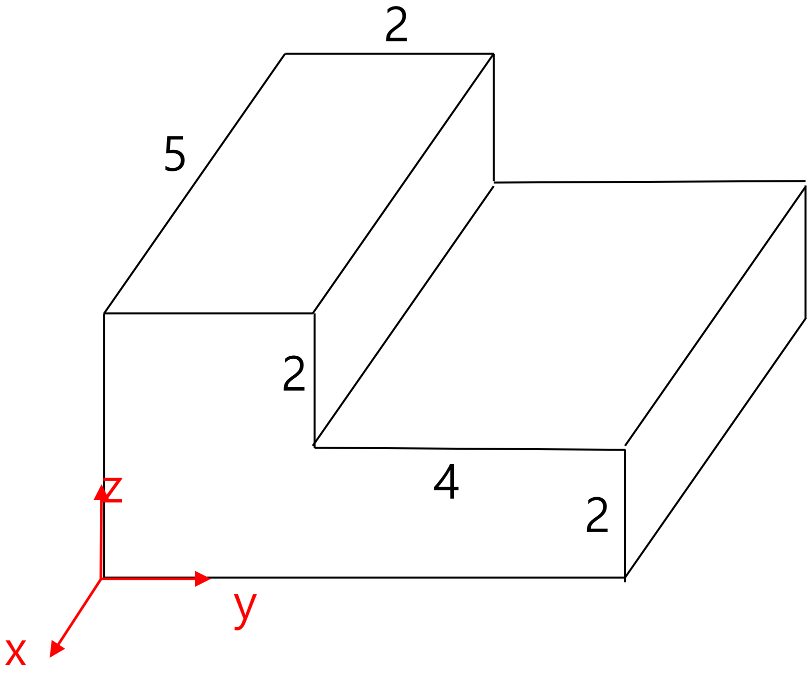



$$F = \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Ex)

$$F = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0.707 & -0.707 & 5 \\ 0 & 0.707 & 0.707 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Examples



# Examples

$$F = \begin{bmatrix} ? & 0 & ? & 5 \\ 0.707 & ? & ? & 3 \\ ? & ? & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
n_x o_x + n_y o_y + n_z o_z &= 0 \quad \text{or} \quad n_x(0) + 0.707(o_y) + n_z(o_z) = 0 \\
n_x a_x + n_y a_y + n_z a_z &= 0 \quad \text{or} \quad n_x(a_x) + 0.707(a_y) + n_z(0) = 0 \\
a_x o_x + a_y o_y + a_z o_z &= 0 \quad \text{or} \quad a_x(0) + a_y(o_y) + 0(o_z) = 0 \\
n_x^2 + n_y^2 + n_z^2 &= 1 \quad \text{or} \quad n_x^2 + 0.707^2 + n_z^2 = 1 \\
o_x^2 + o_y^2 + o_z^2 &= 1 \quad \text{or} \quad 0^2 + o_y^2 + o_z^2 = 1 \\
a_x^2 + a_y^2 + a_z^2 &= 1 \quad \text{or} \quad a_x^2 + a_y^2 + 0^2 = 1
\end{aligned}$$

⇨

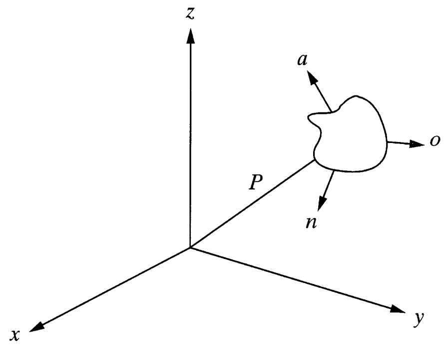
$$\begin{aligned}
0.707\, o_y + n_z o_z &= 0 \\
n_x a_x + 0.707\, a_y &= 0 \\
a_y o_y &= 0 \\
n_x^2 + n_z^2 &= 0.5 \\
o_y^2 + o_z^2 &= 1 \\
a_x^2 + a_y^2 &= 1
\end{aligned}$$

⇨

$$F_1 = \begin{bmatrix} 0.707 & 0 & 0.707 & 5 \\ 0.707 & 0 & -0.707 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad F_2 = \begin{bmatrix} -0.707 & 0 & -0.707 & 5 \\ 0.707 & 0 & -0.707 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Matrix Representation

- Representation of a Rigid Body
  - An object can be represented in space by attaching a frame to it and representing the frame in space

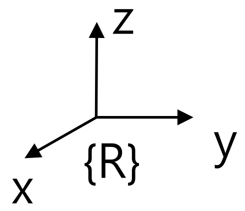


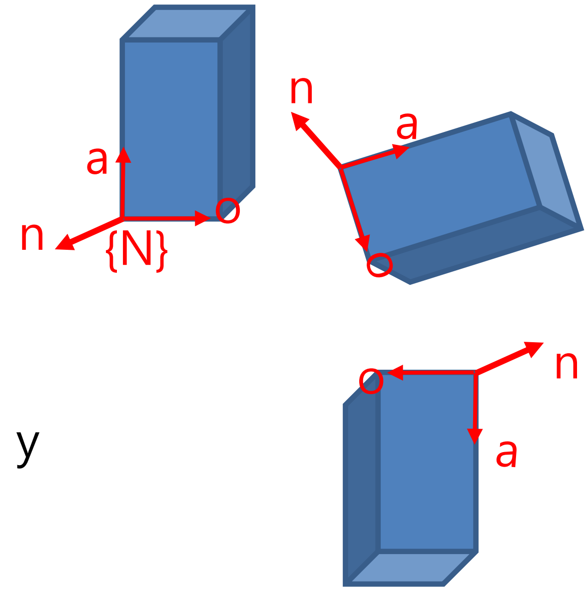

$$F_{object} = \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Homogeneous Transformation Matrix

- Rigid body 의 위치와 자세를 표현하는 4 x 4 행렬

$$F = \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \bar{n} & \bar{o} & \bar{a} & \bar{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$




$$F = {}^{R}T_{N}$$

$\bar{n}$ : {R} 기준 {N} 의 x 축 방향벡터

$\bar{o}$ : {R} 기준 {N} 의 y 축 방향벡터

$\bar{a}$ : {R} 기준 {N} 의 z 축 방향벡터

$\bar{p}$ : {R} 기준 {N} 의 원점 위치벡터

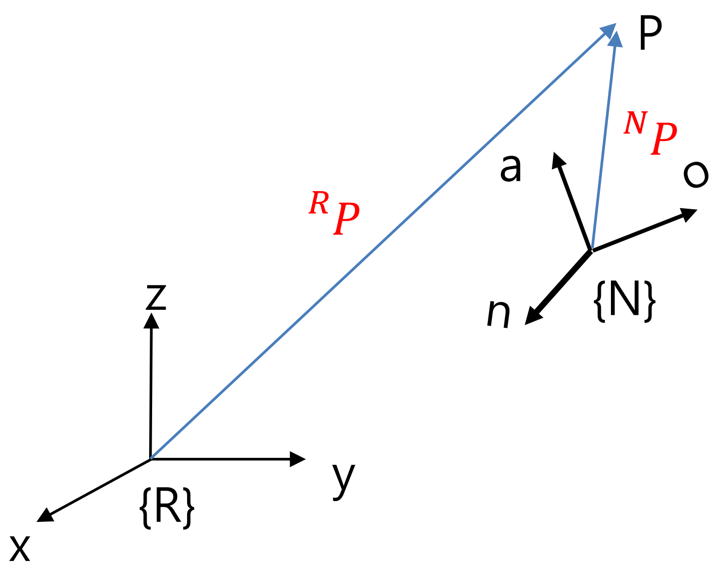
# Homogeneous Transformation Matrix

- Frame 사이의 위치벡터 변환

$$^{R}P = {}^{R}T_{N} \; {}^{N}P$$

$^{R}P$ : {R} frame 기준 위치벡터

$^{N}P$ : {N} frame 기준 위치벡터



# Example

$$ {}^{N}P = (1, 0, 1), \quad {}^{R}T_{N} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \quad {}^{R}P = ? $$

(1) By equation

(2) By drawing

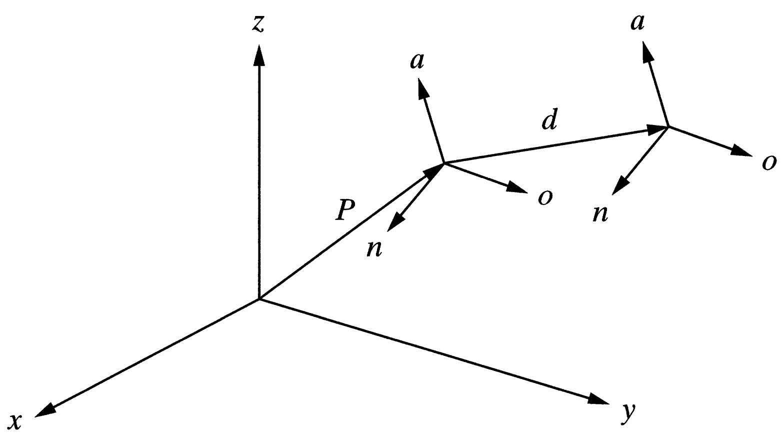
# Representation of Transformations

- A transformation is defined as making a movement in space
  - Pure translation (평행이동)
  - Pure rotation (회전이동)
  - Combination of translations and rotations

$$F_{new} = T\ F_{old}$$

# Representation of Transformations

- **Pure translation**




$$T = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = Trans(d_x, d_y, d_z)$$

(Proof)

$$F_{new} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x + d_x \\ n_y & o_y & a_y & p_y + d_y \\ n_z & o_z & a_z & p_z + d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $$F_{new} = Trans(d_x, d_y, d_z) \times F_{old}$$

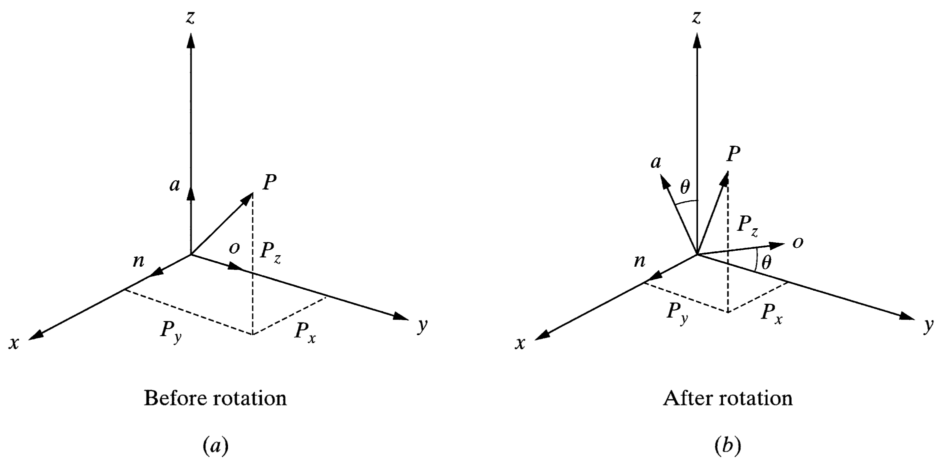
# Example

A frame F has been moved 10 units along the y-axis and 5 units along the z-axis. Find the new frame.

$$F = \begin{bmatrix} 0.527 & -0.574 & 0.628 & 5 \\ 0.369 & 0.819 & 0.439 & 3 \\ -0.766 & 0 & 0.643 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Representation of Transformations

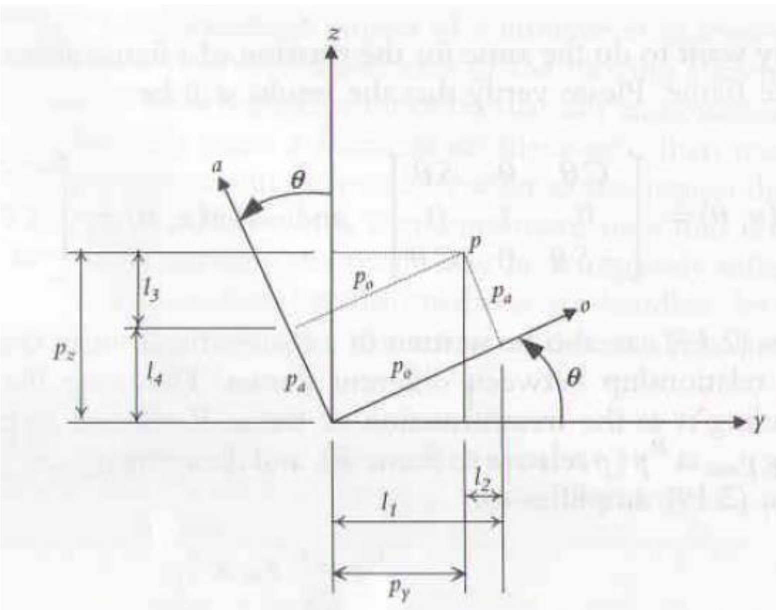
- **Pure Rotation** about an Axis
  - Assumption : The frame is at the origin of the reference frame




$$Rot(x,\theta)=\begin{bmatrix}1&0&0&0\\0&C\theta&-S\theta&0\\0&S\theta&C\theta&0\\0&0&0&1\end{bmatrix}\quad Rot(y,\theta)=\begin{bmatrix}C\theta&0&S\theta&0\\0&1&0&0\\-S\theta&0&C\theta&0\\0&0&0&1\end{bmatrix}\quad Rot(z,\theta)=\begin{bmatrix}C\theta&-S\theta&0&0\\S\theta&C\theta&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

# Representation of Transformations

- $Rot(x, \theta)$




$$
\begin{aligned}
p_x &= p_n \\
p_y &= l_1 - l_2 = p_o \cos\theta - p_a \sin\theta \\
p_z &= l_3 + l_4 = p_o \sin\theta + p_a \cos\theta
\end{aligned}
$$

$$
\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_n \\ p_o \\ p_a \end{bmatrix}
$$

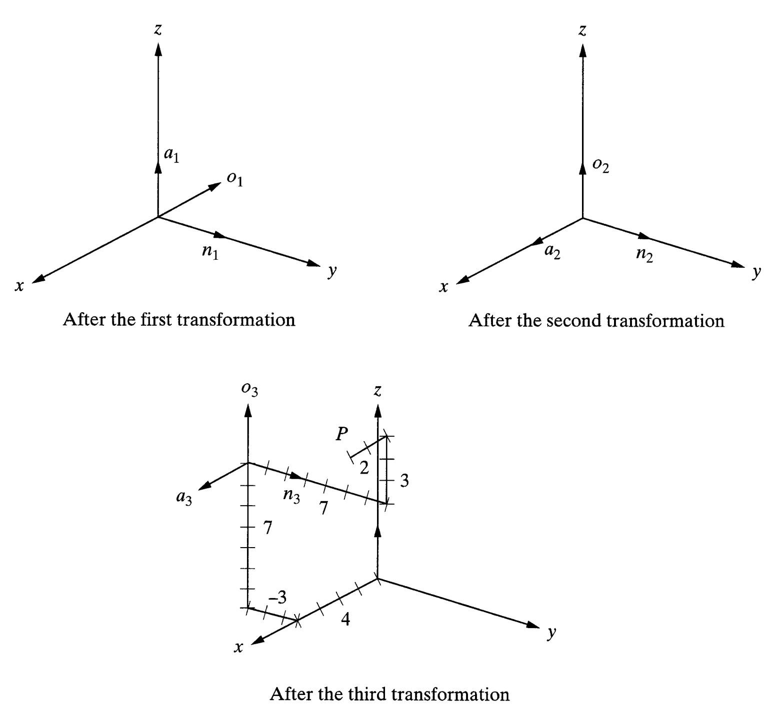
$$
p_{xyz} = Rot(x, \theta) \times p_{noa}
$$

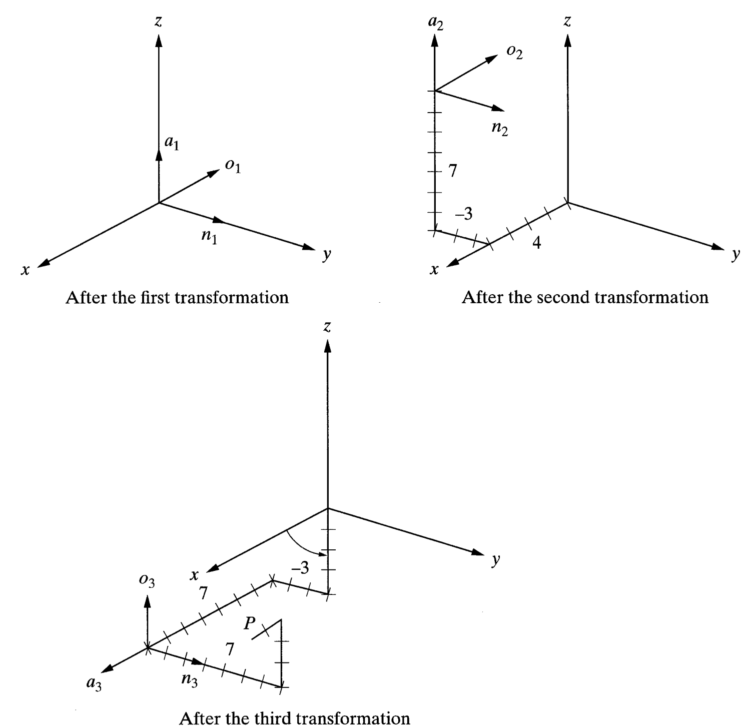
# Example

A point $p(2,3,4)^T$ is attached to a rotating frame. The frame rotates 90° about the $x$-axis of the reference frame. Find the coordinates of the point relative to the reference frame after the rotation, and verify the result graphically.

# Representation of Transformations

- Combined Transformations
  - A number of successive translations and rotations
  - Changing the order of transformations will change the final result




Rot(z,90) →Rot(y,90) →Trans(4,-3,7)

Rot(z,90) →Trans(4,-3,7) →Rot(y,90)

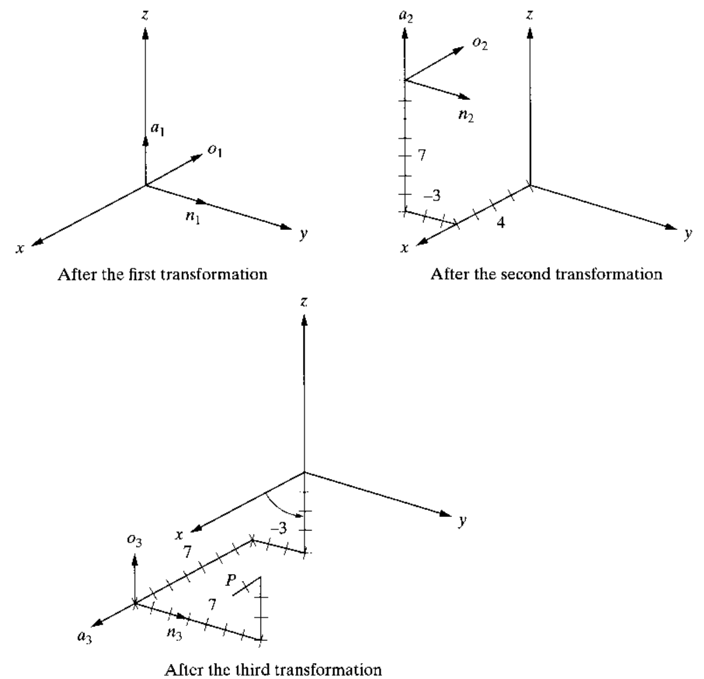
# Representation of Transformations

- Transformations Relative to the Fixed Frame

  ☞ Pre-Multiplication

(ex) P(7,3,1)

- (i) Rot (z, 90)
- (ii) Trans (4,-3,7) along x, y, z axis
- (iii) Rot(y,90)



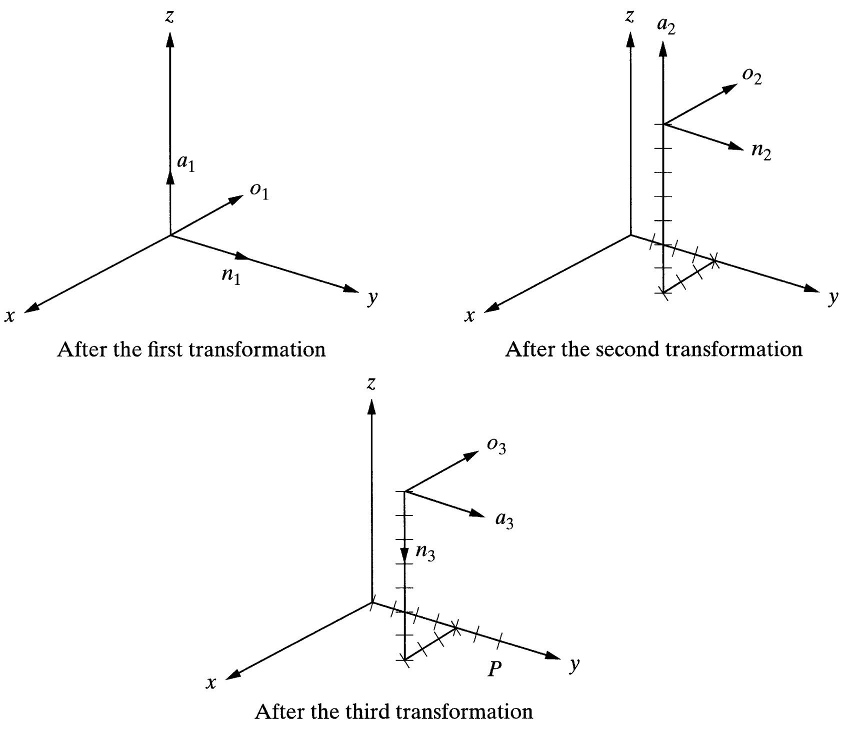
# Representation of Transformations

- Transformations Relative to the Rotating Frame

  ☞ Post-Multiplication

(ex) P(7,3,1)

(i) Rot (a, 90)
(ii) Trans (4,-3,7) along n, o, a axis
(iii) Rot(o,90)




After the first transformation

After the second transformation

After the third transformation

# Representation of Transformations

- Transformations Relative to the Fixed & Rotating Frame

(ex) P(1,5,4)

(i) Rot (x, 90)
(ii) Trans(0,0,3) along n,o,a axis
(iii) Rot(z, 90)
(iv) Trans (0,5,0) along n, o ,a axis

# Inverse of Transformation Matrices

- Inverse transformation

$$T = \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_y & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow T^{-1} = \begin{bmatrix} n_x & n_y & n_z & -\bar{P}\cdot\bar{n} \\ o_x & o_y & o_z & -\bar{P}\cdot\bar{o} \\ a_x & a_y & a_z & -\bar{P}\cdot\bar{a} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$({}^{R}T_{N})^{-1} = {}^{N}T_{R}$$

$$(Rot(X,30^o))^{-1} = Rot(X,-30^o)$$

(Proof)

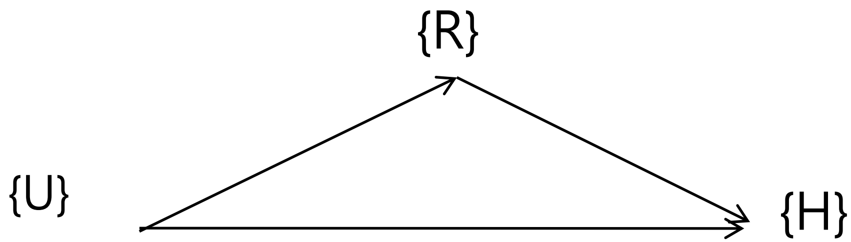
$$T\,T^{-1} = T^{-1}\,T = I$$

(Ex)

$$T = \begin{bmatrix} 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 4 \\ -1 & 0 & 0 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Transformation Equation

(Ex)



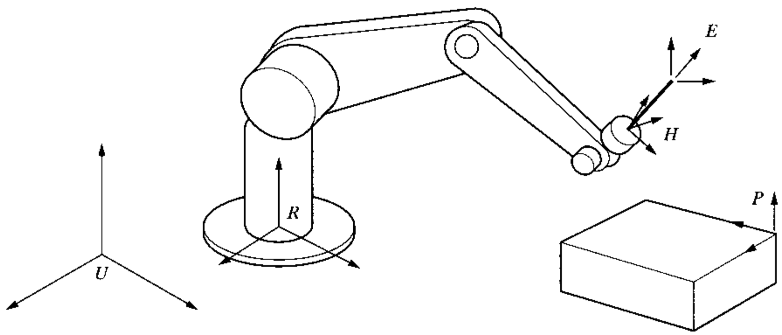

$$^{U}T_{H} = {}^{U}T_{R}\ {}^{R}T_{H} \quad \Rightarrow \quad {}^{R}T_{H} = ({}^{U}T_{R})^{-1}\ {}^{U}T_{H}$$

$$^{U}T_{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{U}T_{R} = \begin{bmatrix} 0 & 0 & -1 & 8 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$^{R}T_{H}$ ?

# Transformation Equation

(Ex)




given ${}^{U}T_{R}$, ${}^{H}T_{E}$, ${}^{U}T_{P}$

find ${}^{R}T_{H}$ (E →P)

# Transformation Equation

(Ex)

Frame definition

{R} robot base, {CAM} hand-eye camera, {H} robot hand

{E} end-effector (or tool) attached at robot hand

{obj} object

Given : ${}^{R}T_{CAM}$, ${}^{R}T_{H}$, ${}^{CAM}T_{obj}$, ${}^{H}T_{E}$

Find : ${}^{E}T_{obj}$